ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/315505798

Euler's Theorem for Homogeneous White Noise Operators

Article *in* Mathematical Physics Analysis and Geometry · March 2017

DOI: 10.1007/s11040-017-9244-2

CITATION
1

READS
167

2 authors:

Abdessatar Barhoumi
King Faisal University
43 PUBLICATIONS **315** CITATIONS

SEE PROFILE

Hafedh Rguigui
Umm Al-Qura University
39 PUBLICATIONS **365** CITATIONS

SEE PROFILE

All content following this page was uploaded by Hafedh Rguigui on 10 February 2019.

The user has requested enhancement of the downloaded file.

Mathematical Physics, Analysis and Geometry manuscript No.
(will be inserted by the editor)

Euler's Theorem For Homogeneous White Noise Operators

Abdessatar Barhoumi · Hafedh Rguigui

Received: date / Accepted: date

Abstract In this paper we introduce a new notion of $\lambda-$order homogeneous operators on the nuclear algebra of white noise operators. Then, we give their Fock expansion in terms of quantum white noise (QWN) fields $\{a_t,\ a_t^*\ ;\ t \in \mathbb{R}\}$. The quantum extension of the scaling transform enables us to prove Euler's theorem in quantum white noise setting.

Keywords QWN-Euler operator, Euler's Theorem, QWN-scaling operator, Homogeneous operator, QWN-derivatives.

Mathematics Subject Classification (2000) 60H40, 46A32, 46F25, 46G20.

1 Introduction and Preliminaries

Let H be the real Hilbert space of square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$ and $E \equiv \mathcal{S}(\mathbb{R})$ be the Schwartz space consisting of rapidly decreasing C^∞-functions. Then, the nuclear Gel'fand triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, dx) \subset \mathcal{S}'(\mathbb{R}) \tag{1}$$

can be reconstructed in a standard way (see Ref. [18]) by the harmonic oscillator $A = 1 + t^2 - d^2/dt^2$ and H. The eigenvalues of A are $2n$, $n = 1, 2, \cdots$, the corresponding eigenfunctions $\{e_n;\ n \geq 1\}$ form an orthonormal basis for $L^2(\mathbb{R})$. In fact (e_n) are the Hermite functions and therefore each e_n is an element of E. The space E is a nuclear space equipped with the Hilbertian norms

$$|\xi|_p = |A^p \xi|_0\,, \qquad \xi \in E, \quad p \in \mathbb{R}$$

A. Barhoumi
Carthage University, Tunisia, Nabeul Preparatory Engineering Institute, Department of Mathematics, Campus Universitaire - Mrezgua - 8000 Nabeul
E-mail: abdessatar.barhoumi@ipein.rnu.tn

H. Rguigui
Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El-Manar, 1060 Tunis, Tunisia
E-mail: hafedh.rguigui@yahoo.fr

and we have

$$E = \operatorname{proj}\lim_{p\to\infty} E_p\,, \qquad E' = \operatorname{ind}\lim_{p\to\infty} E_{-p},$$

where, for $p \geq 0$, E_p is the completion of E with respect to the norm $|\cdot|_p$ and E_{-p} is the topological dual space of E_p. We denote by $N = E + iE$ and $N_p = E_p + iE_p$, $p \in \mathbb{Z}$, the complexifications of E and E_p, respectively. Throughout, we fix a Young function θ satisfying the condition

$$\limsup_{x\to\infty} \frac{\theta(x)}{x^2} < +\infty\cdot \tag{2}$$

Its polar function θ^* is the Young function defined by

$$\theta^*(x) = \sup_{t\geq 0}(tx - \theta(t)), \quad x \geq 0.$$

For more details , see Refs. [8].

For a complex Banach space $(B, \|\cdot\|)$, $\mathcal{H}(B)$ denotes the space of all entire functions on B and for $m > 0$, $\operatorname{Exp}(B, \theta, m)$ is the Banach space

$$\operatorname{Exp}(B, \theta, m) = \Big\{ f \in \mathcal{H}(B);\ \ \|f\|_{\theta,m} := \sup_{z\in B} |f(z)| e^{-\theta(m\|z\|)} < \infty \Big\}.$$

The projective system $\{\operatorname{Exp}(N_{-p}, \theta, m);\ p \in \mathbb{N},\ m > 0\}$ and the inductive system $\{\operatorname{Exp}(N_p, \theta, m);\ p \in \mathbb{N},\ m > 0\}$ give the two nuclear spaces

$$\mathcal{F}_\theta(N') = \operatorname{proj}\lim_{p\to\infty; m\downarrow 0} \operatorname{Exp}(N_{-p}, \theta, m)\,, \qquad \mathcal{G}_\theta(N) = \operatorname{ind}\lim_{p\to\infty; m\to 0} \operatorname{Exp}(N_p, \theta, m). \tag{3}$$

It is noteworthy that, for each $\xi \in N$, the exponential function

$$e_\xi(z) := e^{\langle z,\xi\rangle}, \quad z \in N',$$

belongs to $\mathcal{F}_\theta(N')$ and the set of such test functions spans a dense subspace of $\mathcal{F}_\theta(N')$. In the remainder of this paper we use simply $\mathcal{F}_\theta$ to denote the space $\mathcal{F}_\theta(N')$. The space of continuous linear operators from $\mathcal{F}_\theta$ into its topological dual space $\mathcal{F}_\theta^*$ is denoted by $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and assumed to carry the bounded convergence topology. For $z \in N'$ and $\varphi \in \mathcal{F}_\theta$ with Taylor expansions $\sum_{n=0}^\infty \langle x^{\otimes n}, f_n\rangle$, the holomorphic derivative of φ at $x \in N'$ in the direction z is defined by

$$(a(z)\varphi)(x) := \lim_{\lambda\to 0} \frac{\varphi(x + \lambda z) - \varphi(x)}{\lambda}. \tag{4}$$

We can check that the limit always exists and $a(z) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$. Let $a^*(z) \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta^*)$ be the dual adjoint of $a(z)$, i.e., for $\Phi \in \mathcal{F}_\theta^*$ and $\phi \in \mathcal{F}_\theta$, $\langle\langle a^*(z)\Phi, \phi\rangle\rangle = \langle\langle \Phi, a(z)\phi\rangle\rangle$, where $\langle\langle\cdot,\cdot\rangle\rangle$ denotes the standard bilinear form on $\mathcal{F}_\theta^* \times \mathcal{F}_\theta$. Similarly, for $\psi \in \mathcal{G}_{\theta^*}(N)$ with Taylor expansion $\psi(\xi) = \sum_{n=0}^\infty \langle \psi_n, \xi^{\otimes n}\rangle$ we use the common notation $a(z)\psi$ for the derivative (4) with $z \in N$.

The Wick symbol of $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is by definition [18] a $\mathbb{C}$-valued function on $N \times N$ defined by

$$\sigma(\Xi)(\xi, \eta) = \langle\langle \Xi e_\xi, e_\eta\rangle\rangle e^{-\langle \xi,\eta\rangle}, \quad \xi, \eta \in N. \tag{5}$$

By a density argument, every operator in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is uniquely determined by its Wick symbol. In fact, if $\mathcal{G}_{\theta^*}(N \oplus N)$ denotes the nuclear space obtained as in (3) by replacing N_p by $N_p \times N_p$, we have the following characterization theorem for operator Wick symbols.

Theorem 1 *(See Ref. [13]) The Wick symbol map σ yields a topological isomorphism between $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and $\mathcal{G}_{\theta^*}(N \oplus N)$.*

It is a fundamental fact in quantum white noise theory [18] (see, also Ref. [13]) that every white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ admits a unique Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{6}$$

where, for each pairing $l, m \geq 0$, $\kappa_{l,m} \in (N^{\otimes(l+m)})'_{sym(l,m)}$ and $\Xi_{l,m}(\kappa_{l,m})$ is the integral kernel operator uniquely specified via the Wick symbol transform by

$$\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle, \quad \xi, \eta \in N. \tag{7}$$

For any $S_1,\ S_2 \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, there exists a unique $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, denoted $S_1 \diamond S_2$, such that

$$\sigma(S_1 \diamond S_2) = \sigma(S_1)\sigma(S_2). \tag{8}$$

The operator $S_1 \diamond S_2$ will be referred to as *the Wick product* of S_1 and S_2.

Let θ_n be given by $\theta_n = \inf_{r>0} e^{\theta(r)}/r^n$, $n \in \mathbb{N}$. Then, for $p \in \mathbb{N}$ and $\gamma_1, \gamma_2 > 0$, we define the Hilbert space

$$F_{\theta,\gamma_1,\gamma_2}(N_p \oplus N_p) =$$

$$\Big\{ \overrightarrow{\varphi} = (\varphi_{l,m})_{l,m=0}^{\infty};\ \varphi_{l,m} \in (N_p^{\otimes l} \otimes N_p^{\otimes m})_{sym(l,m)},\ \sum_{l,m=0}^{\infty} (\theta_l \theta_m)^{-2} \gamma_1^{-l} \gamma_2^{-m} |\varphi_{l,m}|_p^2 < \infty \Big\}$$

Put

$$F_\theta(N \oplus N) = \bigcap_{p \in \mathbb{N}, \gamma_1 > 0, \gamma_2 > 0} F_{\theta,\gamma_1,\gamma_2}(N_p \oplus N_p).$$

Theorem 2 *([4]) An operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ if and only if there exists a unique $(\kappa_{l,m})_{l,m} \in F_\theta(N \oplus N)$ such that*

$$\Xi = \Xi_{-\tau} \diamond \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{9}$$

where τ is the usual trace on $N \otimes N$, i.e., $\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle$ and

$$\Xi_{\pm\tau} = \sum_{k=0}^{\infty} \frac{(\pm 1)^k}{k!} \Xi_{k,k}(\tau^{\otimes k}).$$

Let $\mathcal{U}_\theta$ be the space of white noise operators given by

$$\mathcal{U}_\theta = \Big\{ \Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m});\ (\kappa_{l,m})_{l,m} \in F_\theta(N \oplus N) \Big\}.$$

For $x, y \in N$, we put $\kappa_{l,m}(x,y) = \frac{x^{\otimes l}}{l!} \otimes \frac{y^{\otimes m}}{m!}$ and $\Xi^{x,y} := \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}(x,y))$. Then, the set $\{\Xi^{x,y}\,;\ \ x,\, y \in N\}$ spans a dense subspace of $\mathcal{U}_\theta$.

Theorem 3 *([4]) The map f_τ defined by*

$$f_\tau : \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta) \longrightarrow \mathcal{U}_\theta, \quad \Xi \longmapsto \Xi_\tau \diamond \Xi,$$

is a topological isomorphism.

We recall from Ref. [4] the dual pairing: for $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{U}_\theta$ and $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, we define

$$\langle\!\langle \Xi, T \rangle\!\rangle := \sum_{l,m=0}^{\infty} l!m! \langle \kappa_{l,m}, \Phi_{l,m} \rangle.$$

For more details see [4], [5], [6],[22], [23] and [24].

In mathematics, a homogeneous function is a function with multiplicative scaling behavior: if the argument is multiplied by a factor, then the result is multiplied by some power of this factor. More precisely, for $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}\backslash\{0\}$, put $S_t f(x) = f(tx)$, $x \in \mathbb{R}^d$. For a given $\lambda \in \mathbb{R}$, an element $f \in L^2(\mathbb{R}^d)$ is said to be $\lambda-$order homogeneous if $S_t f(x) = t^\lambda f(x)$ for each $t \in \mathbb{R}\backslash\{0\}$ and $x \in \mathbb{R}^d$. It is well known that f is $\lambda-$order homogeneous if and only if it satisfies the so-called Euler equation

$$\sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} f = \lambda f \cdot \tag{10}$$

In infinite dimension analysis, an analogue of the Euler operator $\displaystyle\sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i}$ was introduced in [17] as follows

$$\Delta_E = \sum_{i=1}^{\infty} (a^*(e_i) + a(e_i)) a(e_i) = \sum_{i=1}^{\infty} \langle \,\cdot\, , e_i \rangle a(e_i).$$

Moreover, the scaling transformation S_t is defined at $\varphi(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, \varphi_n \rangle \in \mathcal{F}_\theta$ by

$$S_t \varphi(x) = \sum_{n=0}^{\infty} \left\langle x^{\otimes n}, \frac{t^n}{n!} \sum_{l=0}^{\infty} (t^2 - 1)^l \frac{(n+2l)!}{l! 2^l} \tau^{\otimes l} \widehat{\otimes}_l \varphi_{n+2l} \right\rangle, \quad x \in N' \cdot$$

For $\lambda \in \mathbb{R}$, φ is said to be $\lambda-$order homogeneous if $S_t \varphi = t^\lambda \varphi$ for any $t \in \mathbb{R}\backslash\{0\}$. It is proved in [20] that φ is $\lambda-$order homogeneous if and only if it satisfies the Euler equation

$$\Delta_E \varphi = \lambda \varphi \cdot \tag{11}$$

The main purpose of this paper is the study of the `QWN`-analogue of (11). We start by introducing a `QWN`-Scaling transformation and a `QWN`-second quantization. These transformations will be used to introduce the notion of $\lambda-$order homogeneous operators. Then, as a first main result we give their Fock expansions (see Theorem 5). Our second main result is stated in Theorem 7, where we show that a white noise operator Ξ is $\lambda-$order homogeneous if and only if it satisfies the following `QWN`-Euler equation

$$\Delta_E^Q \Xi = \lambda \Xi.$$

Here Δ_E^Q is the `QWN`-Euler operator defined in [6].

2 Fundamental QWN-Operators

2.1 QWN-Laplacians

From [6], the QWN-Gross Laplacian and QWN-conservation operator can be defined through Theorem 3 on $\mathcal{U}_\theta$, respectively, by

$$\Delta_G^Q = \sum_{j=1}^{\infty} D_{e_j}^{+} D_{e_j}^{+} + \sum_{j=1}^{\infty} D_{e_j}^{-} D_{e_j}^{-},$$

$$N^Q = \sum_{j=1}^{\infty} (D_{e_j}^{+})^{*} (D_{e_j})^{+} + \sum_{j=1}^{\infty} (D_{e_j}^{-})^{*} D_{e_j}^{-},$$

where, for $\zeta \in N$,

$$D_\zeta^{+} \Xi = [a(\zeta), \Xi], \qquad D_\zeta^{-} \Xi = -[a^{*}(\zeta), \Xi] \tag{12}$$

are the *creation derivative* and *annihilation derivative* of Ξ, (see [12]).

Lemma 1 *For any $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$, we have*

$$N^Q \Xi = \sum_{l,m=0}^{\infty} (l+m) \Xi_{l,m}(\kappa_{l,m}). \tag{13}$$

Proof From [6], we have, for $x, y \in N$

$$\sigma(N^Q \Xi^{x,y})(\xi, \eta) = (\langle x, \eta\rangle + \langle y, \xi\rangle) \sigma(\Xi^{x,y})(\xi, \eta).$$

On the other hand, denoting the right hand side of (13) by A^Q, we get

$$\sigma(A^Q \Xi^{x,y})(\xi, \eta)$$

$$\begin{aligned} &= \sum_{l=1}^{\infty} \sum_{m=0}^{\infty} \langle x, \eta\rangle \frac{\langle x, \eta\rangle^{l-1}}{(l-1)!} \frac{\langle y, \xi\rangle^{m}}{m!} + \sum_{l=0}^{\infty} \sum_{m=1}^{\infty} \frac{\langle x, \eta\rangle^{l}}{l!} \langle y, \xi\rangle \frac{\langle y, \xi\rangle^{m-1}}{(m-1)!} \\ &= (\langle x, \eta\rangle + \langle y, \xi\rangle) \sigma(\Xi^{x,y})(\xi, \eta). \end{aligned}$$

Then, by a density argument we complete the proof.

It is noteworthy that the identity (13) holds true for $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^{*})$.

Proposition 1 *Let $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^{*})$. Then, we have*

$$(\Delta_G^Q)^{*} T = \{\Xi_{2,0}(\tau) + \Xi_{0,2}(\tau)\} \diamond T \tag{14}$$

$$(N^Q)^{*} T = N^Q T. \tag{15}$$

Proof From [1], for $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$, we have

$$\Delta_G^Q \Xi = \sum_{l,m=0}^{\infty} (l+2)(l+1)\Xi_{l,m}(\tau \otimes^2 \kappa_{l+2,m}) + \sum_{l,m=0}^{\infty} (m+2)(m+1)\Xi_{l,m}(\kappa_{l,m+2} \otimes_2 \tau), \tag{16}$$

where, for $z_p \in (N^{\otimes p})'$, and $\xi_{l+m-p} \in N^{\otimes(l+m-p)}$, $p \leq l+m$, the contractions $z_p \otimes_p \kappa_{l,m}$ and $\kappa_{l,m} \otimes^p z_p$ are defined by

$$\langle z_p \otimes^p \kappa_{l,m}, \xi_{l-p+m} \rangle = \langle \kappa_{l,m}, z_p \otimes \xi_{l-p+m} \rangle,$$

$$\langle \kappa_{l,m} \otimes_p z_p, \xi_{l+m-p} \rangle = \langle \kappa_{l,m}, \xi_{l+m-p} \otimes z_p \rangle.$$

Then, for $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, we obtain
$\langle\!\langle T, \Delta_G^Q \Xi \rangle\!\rangle$

$$\begin{aligned}
&= \sum_{l,m=0}^{\infty} l!m!(l+2)(l+1)\langle \Phi_{l,m}, \tau \otimes^2 \kappa_{l+2,m} \rangle \\
&\quad + \sum_{l,m=0}^{\infty} l!m!(m+2)(m+1)\langle \Phi_{l,m}, \kappa_{l,m+2} \otimes_2 \tau \rangle \\
&= \sum_{l=2}^{\infty}\sum_{m=0}^{\infty} l!m!\langle \tau \otimes \Phi_{l-2,m}, \kappa_{l,m} \rangle + \sum_{l=0}^{\infty}\sum_{m=2}^{\infty} l!m!\langle \Phi_{l,m-2} \otimes \tau, \kappa_{l,m} \rangle.
\end{aligned}$$

Therefore, we get

$$(\Delta_G^Q)^* T = \sum_{l=2}^{\infty}\sum_{m=0}^{\infty} \Xi_{l,m}(\tau \otimes \Phi_{l-2,m}) + \sum_{l=0}^{\infty}\sum_{m=2}^{\infty} \Xi_{l,m}(\Phi_{l,m-2} \otimes \tau),$$

which yields

$$\begin{aligned}
\sigma((\Delta_G^Q)^* T)(\xi, \eta) &= \sum_{l=2}^{\infty}\sum_{m=0}^{\infty} \langle \tau \otimes \Phi_{l-2,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \\
&\quad + \sum_{l=0}^{\infty}\sum_{m=2}^{\infty} \langle \Phi_{l,m-2} \otimes \tau, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \\
&= \{\langle \eta, \eta \rangle + \langle \xi, \xi \rangle\} \sigma(T)(\xi, \eta) \\
&= \sigma(\Xi_{2,0}(\tau) + \Xi_{0,2}(\tau))(\xi, \eta)\sigma(T)(\xi, \eta).
\end{aligned}$$

This gives

$$(\Delta_G^Q)^* T = \{\Xi_{2,0}(\tau) + \Xi_{0,2}(\tau)\} \diamond T$$

as desired. (15) follows from (13).

2.2 QWN-Second Quantization

We start by clarifying the topology of the nuclear algebra $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. From Theorem 1, we have the topological isomorphism:

$$\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*) \simeq \mathcal{G}_{\theta^*}(N \oplus N) = \bigcup_{p\geq 0, \gamma>0} \mathrm{Exp}(N_p \oplus N_p, \theta^*, \gamma).$$

For $p \geq 0$ and $\gamma > 0$, let $\mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ denotes the subspace of all $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ which correspond to elements in $\mathrm{Exp}(N_p \oplus N_p, \theta^*, \gamma)$. The topology of $\mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is naturally induced from the norm of the Banach space $\mathrm{Exp}(N_p \oplus N_p, \theta^*, \gamma)$ which will be denoted by $\| \cdot \|_{\theta,-p,\gamma}$, i.e., for $\Xi \in \mathcal{L}_{\theta,-p,\gamma}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$,

$$\|\Xi\|_{\theta,-p,\gamma} = \|\sigma \Xi\|_{\theta^*,-p,\gamma} = \sup_{\xi,\eta\in N_p} |\sigma(\Xi)(\xi,\eta)|\, e^{-\theta^*(\gamma|\xi|_p)-\theta^*(\gamma|\eta|_p)}.$$

For $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and $t \in \mathbb{R}$, we define the operator $\Gamma^Q(t)$ by

$$\Gamma^Q(t)\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(t^{l+m}\Phi_{l,m}). \tag{17}$$

We denote by $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ the group of all linear homeomorphisms from $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ onto itself.

Proposition 2 *$\{\Gamma^Q(e^t)\}_{t\in\mathbb{R}}$ is a regular one-parameter subgroup of $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ with infinitesimal generator N^Q.*

Proof The proof of the fact that $\{\Gamma^Q(e^t)\}_{t\in\mathbb{R}}$ is a one-parameter subgroup of $GL(\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*))$ is straightforward. Since we have

$$e^{t(l+m)} = 1 + t(l+m) + t^2 \sum_{k=0}^\infty \frac{t^k}{(k+2)!}(l+m)^{(k+2)}$$

then, for $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, one can write

$$\Gamma^Q(e^t)\Xi = \Xi + tN^Q(\Xi) + t^2 \Lambda(t)(\Xi) \tag{18}$$

where

$$\Lambda(t)(\Xi) = \sum_{l,m=0}^\infty \Xi_{l,m}(\Lambda_{l,m}(t)\Phi_{l,m})$$

with

$$\Lambda_{l,m}(t) = \sum_{k=0}^\infty \frac{t^k}{(k+2)!}(l+m)^{(k+2)}.$$

Now, for $|t| \leq 1$, using a similar computation as in [7], one can show that, there exist $c, r, r' > 0$ and $p, q \geq 0$ such that

$$\left\| \frac{\sigma(\Gamma^Q(e^t)\Xi) - \sigma(\Xi)}{t} - \sigma(N^Q \Xi) \right\|_{\theta^*,-p,r'} \leq c|t| \|\sigma(\Xi)\|_{\theta^*,-q,r}.$$

It then follows

$$\lim_{t\to 0} \sup_{\|\sigma(\Xi)\|_{\theta^*,-q,r}\leq 1} \left\| \frac{\sigma(\Gamma^Q(e^t)\Xi) - \sigma(\Xi)}{t} - \sigma(N^Q T) \right\|_{\theta^*,-p,\gamma} = 0.$$

This proves the desired statement.

2.3 QWN-Scaling Transformation

Motivated by the classical case studied in [17] and [20], we define the QWN-scaling transformation acting on $\Xi=\sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})\in\mathcal{U}_\theta$ by

$$S_t^Q(\Xi):=$$

$$\sum_{j,k,l,m=0}^{\infty} t^{l+m}(t^2-1)^{j+k}\frac{(l+2j)!(m+2k)!}{2^{j+k}j!k!l!m!}\Xi_{l,m}(\tau^{\otimes j}\otimes^{2j}\kappa_{l+2j,m+2k}\otimes_{2k}\tau^{\otimes k}) \qquad (19)$$

We recall from Ref. [6] and Theorem 3 that the QWN-Fourier-Gauss transform $G^Q_{K_1,K_2;B_1,B_2}$ is a continuous linear operator from $\mathcal{U}_\theta$ into itself defined by

$$G^Q_{K_1,K_2;B_1,B_2}\Xi=\sum_{l,m}^{\infty}\Xi_{l,m}(g_{l,m}) \qquad (20)$$

where $K_i,\ B_i\in\mathcal{L}(N',N')\cap\mathcal{L}(N,N)$, $i=1,2$ and $g_{l,m}$ is given by

$$g_{l,m}=\sum_{j,k=0}^{\infty}\frac{(l+2j)!(m+2k)!}{l!m!j!k!}\left(B_1^{\otimes l}\otimes B_2^{\otimes m}\right)\left(\tau_{K_1}^{\otimes j}\otimes^{2j}\kappa_{l+2j,m+2k}\otimes_{2k}\tau_{K_2}^{\otimes k}\right).$$

In our setting, we observe that $S_t^Q=G^Q_{\frac{1}{2}(t^2-1)I,\frac{1}{2}(t^2-1)I;tI,tI}$.

Theorem 4 *Let $\Xi=\sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})\in\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta^*)$. Then, $(S_t^Q)^*\Xi$ is given by*

$$(S_t^Q)^*\Xi=F(t)\diamond\Gamma^Q(t)(\Xi),$$

where $F(t)$ is given by

$$F(t)=\sum_{j,k=0}^{\infty}\frac{(t^2-1)^{j+k}}{2^{j+k}j!k!}\Xi_{2j,2k}(\tau^{\otimes j}\otimes\tau^{\otimes k}). \qquad (21)$$

Proof For $\Xi=\sum_{l,m=0}\Xi_{l,m}(\kappa_{l,m})\in\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}_\theta^*)$ and $T=\sum_{l,m=0}\Xi_{l,m}(\Phi_{l,m})\in\mathcal{U}_\theta$, we have

$$\langle\!\langle\Xi,S_t^QT\rangle\!\rangle$$

$$=\sum_{j,k,l,m=0}^{\infty}l!m!\Big\langle\kappa_{l,m},(t^2-1)^{j+k}t^{l+m}\frac{(l+2j)!(m+2k)!}{2^{j+k}j!l!}\tau^{\otimes j}\otimes^{2j}\Phi_{l+2j,m+2k}\otimes_{2k}\tau^{\otimes k}\Big\rangle$$

$$=\sum_{p,q=0}^{\infty}p!q!\Big\langle\sum_{j=0}^{[p/2]}\sum_{k=0}^{[q/2]}\frac{(t^2-1)^{j+k}}{2^{j+k}j!k!}t^{p+q-2j-2k}\tau^{\otimes j}\otimes\kappa_{p-2j,q-2k}\otimes\tau^{\otimes k},\Phi_{p,q}\Big\rangle.$$

This yields

$$(S_t^Q)^*\Xi=\sum_{p,q=0}^{\infty}\sum_{j=0}^{[p/2]}\sum_{k=0}^{[q/2]}\frac{t^{p+q-2j-2k}(t^2-1)^{j+k}}{2^{j+k}j!k!}\Xi_{p,q}(\tau^{\otimes j}\otimes\kappa_{p-2j,q-2k}\otimes\tau^{\otimes k}). \qquad (22)$$

On the other hand, we have

$$
\begin{aligned}
&\sigma(F(t) \diamond \Gamma^Q(t)(\Xi))(\xi,\eta) \\
&\quad = \sigma(F(t))(\xi,\eta)\sigma(\Gamma^Q(t)(\Xi))(\xi,\eta) \\
&\quad = \sum_{j,k,l,m=0}^{\infty} \frac{t^{l+m}(t^2-1)^{j+k}}{2^{j+k}j!k!} \left\langle \tau^{\otimes j} \otimes \kappa_{l,m} \otimes \tau^{\otimes k}, \eta^{\otimes l+2j} \otimes \xi^{\otimes m+2k} \right\rangle \\
&\quad = \sum_{p,q=0}^{\infty} \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{t^{p+q-2j-2k}(t^2-1)^{j+k}}{2^{j+k}j!k!} \left\langle \tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}, \eta^{\otimes p} \otimes \xi^{\otimes q} \right\rangle
\end{aligned}
$$

or equivalently

$$
F(t) \diamond \Gamma^Q(t)(\Xi)
$$

$$
= \sum_{p,q=0}^{\infty} \sum_{j=0}^{[p/2]} \sum_{k=0}^{[q/2]} \frac{t^{p+q-2j-2k}(t^2-1)^{j+k}}{2^{j+k}j!k!} \Xi_{p,q}(\tau^{\otimes j} \otimes \kappa_{p-2j,q-2k} \otimes \tau^{\otimes k}).
$$

Comparing with (22), the statement follows.

Remark 1 Using (19), for $x, y \in N$, we have

$$
S_t^Q(\Xi^{x,y}) = \exp\big\{\frac{1}{2}(t^2-1)\langle x,x\rangle + \frac{1}{2}(t^2-1)\langle y,y\rangle\big\}\Xi^{tx,ty}. \tag{23}
$$

Then, for all $s, t \in \mathbb{R}\backslash\{0\}$, by a density argument, one can verify that

$$
S_s^Q S_t^Q = S_{st}^Q. \tag{24}
$$

In particular, for all $s \in \mathbb{R}$ and $t \in \mathbb{R}\backslash\{0\}$, we get

$$
S_{1+\frac{s}{t}}^Q S_t^Q = S_{s+t}^Q. \tag{25}
$$

3 Euler's Theorem For Homogeneous Operator

3.1 Homogeneous Operator

Definition 1 Let $\Xi \in \mathcal{U}_\theta$ and $\lambda \in \mathbb{R}$. We say that Ξ is λ-order homogeneous if for each $t \in \mathbb{R}\backslash\{0\}$ we have

$$
S_t^Q(\Xi) = t^\lambda \Xi. \tag{26}
$$

This definition is motivated by the classical case studied in [17].

Lemma 2 *Let $l,m \geq 0$ and $\kappa_{l,m} \in (N^{\otimes l} \otimes N^{\otimes m})_{sym(l,m)}$. Then, for K_i, $B_i \in \mathcal{L}(N', N') \cap \mathcal{L}(N, N)$, $i = 1, 2$, we have*

$$
G^Q_{K_1,K_2;B_1,B_2}\Xi_{l,m}(\kappa_{l,m}) = \sum_{p=0}^{[l/2]} \sum_{q=0}^{[m/2]} \Xi_{l-2p,m-2q}(g_{l-2p,m-2q}), \tag{27}
$$

where $g_{l-2p,m-2q}$ is given by

$$
g_{l-2p,m-2q} =
$$

$$
\frac{l!m!}{(l-2p)!(m-2q)!p!q!} \left(B_1^{\otimes(l-2p)} \otimes B_2^{\otimes(m-2q)}\right) \left(\tau_{K_1}^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau_{K_2}^{\otimes q}\right) \tag{28}
$$

and τ_{K_i} is the K_i-trace defined by $\langle \tau_{K_i}, z \otimes w\rangle = \langle K_i z, w\rangle$.

Proof The operator $\Xi_{l,m}(\kappa_{l,m})$ can be rewritten as

$$\Xi_{l,m}(\kappa_{l,m}) = \sum_{\alpha,\beta=0}^{\infty} \Xi_{\alpha,\beta}(f_{\alpha,\beta}),$$

where $f_{\alpha,\beta}$ is defined by

$$f_{\alpha,\beta} = \begin{cases} \kappa_{l,m} & if \ \ (l,m) = (\alpha,\beta) \\ 0 & if \ \ (l,m) \neq (\alpha,\beta). \end{cases} \tag{29}$$

Then, by using (20), we get

$$G^{Q}_{K_1,K_2;B_1,B_2}\Xi_{l,m}(\kappa_{l,m}) = \sum_{\alpha,\beta=0}^{\infty} \Xi_{\alpha,\beta}(g_{\alpha,\beta}),$$

with

$$g_{\alpha,\beta} = \sum_{j,k=0}^{\infty} \frac{(\alpha+2j)!(\beta+2k)!}{\alpha!\beta!j!k!} \left(B_1^{\otimes\alpha} \otimes B_2^{\otimes\beta}\right) \left(\tau_{K_1}^{\otimes j} \otimes^{2j} f_{\alpha+2j,\beta+2k} \otimes_{2k} \tau_{K_2}^{\otimes k}\right).$$

From (29), we observe that $g_{\alpha,\beta} = 0$ for $\alpha > l$ or $\beta > m$. Thus, we obtain

$$G^{Q}_{K_1,K_2;B_1,B_2}\Xi = \sum_{0\le\alpha\le l} \sum_{0\le\beta\le m} \Xi_{\alpha,\beta}(g_{\alpha,\beta}),$$

with

$$g_{\alpha,\beta} = \sum_{j,k=0}^{\infty} \sum_{2j=l-\alpha} \sum_{2k=m-\beta} \frac{l!m!}{\alpha!\beta!j!k!} \left(B_1^{\otimes\alpha} \otimes B_2^{\otimes\beta}\right) \left(\tau_{K_1}^{\otimes j} \otimes^{2j} \kappa_{l,m} \otimes_{2k} \tau_{K_2}^{\otimes k}\right).$$

Moreover, when $l-\alpha = 2p+1$ or $m-\beta = 2q+1$, we have $g_{\alpha,\beta} = 0$. The case $l-\alpha = 2p$ and $m-\beta = 2q$ gives

$$g_{\alpha,\beta} = \frac{l!m!}{\alpha!\beta!p!q!} \left(B_1^{\otimes\alpha} \otimes B_2^{\otimes\beta}\right) \left(\tau_{K_1}^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau_{K_2}^{\otimes q}\right).$$

Replacing α by $l-2p$ and β by $m-2q$, we get the desired statement.

The following theorem gives the Fock expansion of the $\lambda-$order homogeneous operator in $\mathcal{U}_\theta$.

Theorem 5 *Let $\lambda \in \mathbb{N}$ and $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$. Then, Ξ is λ-order homogeneous if and only if*

$$\begin{aligned}\Xi = \sum_{l=0}^{\lambda} \sum_{p=0}^{[l/2]} \sum_{q=0}^{[\frac{\lambda-l}{2}]} \int_{\mathbb{R}^{\lambda-2p-2q}} & \Upsilon_{l-2p,\lambda-l-2q}(s_1,\cdots,s_{l-2p},t_1,\cdots,t_{\lambda-l-2q}) \\ & a^*_{s_1}\cdots a^*_{s_{l-2p}} a_{t_1}\cdots a_{t_{\lambda-l-2q}} ds_1\cdots ds_{l-2p} dt_1\cdots dt_{\lambda-l-2q},\end{aligned}$$

where $\Upsilon_{l-2p,\lambda-l-2q}$ is given by

$$\Upsilon_{l-2p,\lambda-l-2q} =$$
$$\sum_{j,k=0}^{\infty} \frac{(l+2j)!(\lambda-l+2k)!(-1)^{j+k}}{j!k!p!q!(l-2p)!(\lambda-l-2q)!2^{j+k}2^{p+q}} \left(\tau^{\otimes(j+p)} \otimes^{2(j+p)} \kappa_{l+2j,\lambda-l+2k} \otimes_{2(k+q)} \tau^{\otimes(k+q)}\right).$$

Proof In the following we set

$$\mathcal{G}^Q := G^Q_{-\frac{1}{2}I,-\frac{1}{2}I;-iI,-iI}.$$

Motivated by the classical case (see [16]), we can show that $\mathcal{G}^Q$ is a topological isomorphism from $\mathcal{U}_\theta$ into itself. Moreover,

$$(\mathcal{G}^Q)^{-1}\Xi = G^Q_{-\frac{1}{2}I,-\frac{1}{2}I;iI,iI}\Xi. \tag{30}$$

For any $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$ and $t \in \mathbb{R}$, the technical identity

$$S_t^Q(\Xi) = (\mathcal{G}^Q)^{-1}\Gamma^Q(t)\mathcal{G}^Q(\Xi) \tag{31}$$

holds true. Indeed, by direct computation, we have

$$\mathcal{G}^Q\Xi^{x,y} = \exp\big\{-\frac{1}{2}\langle x,x\rangle - \frac{1}{2}\langle y,y\rangle\big\}\Xi^{-ix,-iy}, \quad x,y \in N.$$

Therefore,

$$\Gamma^Q(t)\mathcal{G}^Q(\Xi^{x,y}) = \exp\big\{-\frac{1}{2}\langle x,x\rangle - \frac{1}{2}\langle y,y\rangle\big\}\Xi^{-itx,-ity}.$$

Then, we obtain

$$(\mathcal{G}^Q)^{-1}\Gamma^Q(t)\mathcal{G}^Q(\Xi^{x,y}) = \exp\big\{\frac{1}{2}(t^2-1)(\langle x,x\rangle + \langle y,y\rangle)\big\}\Xi^{tx,ty}.$$

Hence, by (23) we deduce that

$$S_t^Q(\Xi^{x,y}) = (\mathcal{G}^Q)^{-1}\Gamma^Q(t)\mathcal{G}^Q(\Xi^{x,y}),$$

which proves (31) by density argument.

In view of (31), equation (26) can be rewritten as follows

$$\Gamma^Q(t)\mathcal{G}^Q(\Xi) = t^\lambda \mathcal{G}^Q(\Xi). \tag{32}$$

Let $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m}) \in \mathcal{U}_\theta$ be the unique Fock expansion of the operator $T = \mathcal{G}^Q(\Xi)$, where $\Phi_{l,m}$ is given by

$$\Phi_{l,m} = \sum_{j,k=0}^{\infty} \frac{(l+2j)!(m+2k)!(-i)^{l+m}(-1)^{j+k}}{l!m!j!k!2^{j+k}} \left(\tau^{\otimes j} \otimes^{2j} \Phi_{l+2j,m+2k} \otimes_{2k} \tau^{\otimes k}\right).$$

Then (32) can be rewritten as

$$\Gamma^Q(t)T = t^\lambda T, \tag{33}$$

or equivalently

$$\sum_{l,m=0}^{\infty} t^{l+m}\Xi_{l,m}(\Phi_{l,m}) = \sum_{l,m=0}^{\infty} t^\lambda \Xi_{l,m}(\Phi_{l,m}).$$

From the uniqueness of the Fock expansion, this last equation is satisfied if and only if $\lambda = l + m$. Then, T satisfies (33) if and only if

$$T = \sum_{l=0}^{\lambda} \Xi_{l,\lambda-l}(\Phi_{l,\lambda-l}).$$

Therefore, by Eq. (27), we obtain

$$\begin{aligned} \Xi &= (\mathcal{G}^Q)^{-1} T \\ &= \sum_{l=0}^{\lambda} G^Q_{-\frac{1}{2}I,-\frac{1}{2}I;iI,iI}(\Xi_{l,\lambda-l}(\Phi_{l,\lambda-l})) \\ &= \sum_{l=0}^{\lambda} \sum_{p=0}^{[l/2]} \sum_{q=0}^{[\frac{\lambda-l}{2}]} \Xi_{l-2p,\lambda-l-2q}(\Upsilon_{l-2p,\lambda-l-2q}) \end{aligned}$$

where $\Upsilon_{l-2p,\lambda-l-2q}$ is given by

$$\Upsilon_{l-2p,\lambda-l-2q} = \frac{l!(\lambda-l)!i^{\lambda}}{(l-2p)!(\lambda-l-2p)!p!q!2^{p+q}} \left(\tau^{\otimes p} \otimes^{2p} \Phi_{l,\lambda-l} \otimes_{2q} \tau^{\otimes_q}\right)$$

and

$$\Phi_{l,\lambda-l} = \sum_{j,k=0}^{\infty} \frac{(l+2j)!(\lambda-l+2k)!(-1)^{\lambda}(-1)^{j+k}}{l!(\lambda-l)!j!k!2^{j+k}} \left(\tau^{\otimes j} \otimes^{2j} \kappa_{l+2j,\lambda-l+2k} \otimes_{2k} \tau^{\otimes k}\right).$$

Then, we get

$$\begin{aligned} \Upsilon_{l-2p,\lambda-l-2q} = & \sum_{j,k=0}^{\infty} \frac{(l+2j)!(\lambda-l+2k)!(-1)^{j+k}}{j!k!p!q!(l-2p)!(\lambda-l-2q)!2^{j+k}2^{p+q}} \\ & \times \left(\tau^{\otimes(j+p)} \otimes^{2(j+p)} \kappa_{l+2j,\lambda-l+2k} \otimes_{2(k+q)} \tau^{\otimes(k+p)}\right) \end{aligned}$$

as desired.

Example 1 (The 1-order homogeneous operators). For $z, w \in N$, the operator

$$\Xi = a^*(z) + a(w)$$

is a 1-order homogeneous operator. In particular, for $z = w$, the multiplication operator $\Xi = M_{\langle \cdot, z\rangle}$ is 1-order homogeneous.

Example 2 (The 2-order homogeneous operators). For $\kappa_{0,2}$, $\kappa_{2,0}$, $\kappa_{1,1} \in N \otimes N$, the operators

$$\begin{aligned} &\Xi_{2,0}(\kappa_{2,0}) + \Xi_{0,0}(\langle \tau, \kappa_{2,0}\rangle), \\ &\Xi_{0,2}(\kappa_{0,2}) + \Xi_{0,0}(\langle \tau, \kappa_{0,2}\rangle), \\ &\Xi_{1,1}(\kappa_{1,1}) \end{aligned}$$

are 2-order homogeneous. Note that if we take $\kappa_{0,2} = \kappa_{2,0} = \tau_K$ for $K \in \mathcal{L}(N', N)$ such that $\langle \tau, \tau_K\rangle = 0$, then the $K-$Gross Laplacian $\Delta_G(K) = \Xi_{2,0}(\tau_K)$ and its dual $\Delta^*_G(K) = \Xi_{0,2}(\tau_K)$ are 2-order homogeneous operators. Moreover, for $B \in \mathcal{L}(N', N)$, the conservation operator $N(B) = \Xi_{1,1}(\tau_B)$ is 2-order homogeneous operator.

Remark 2 Let $\lambda \in \mathbb{N}$. Then, using Theorem 3, $\Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ is λ-order homogeneous if and only if

$$\Xi = \Xi_{-\tau} \diamond \Xi_h$$

where Ξ_h is λ-order homogeneous in $\mathcal{U}_\theta$.

3.2 Euler's Theorem For Homogeneous Operator

From Ref. [6], the `QWN`-Euler operator can be represented, via Theorem 3, as a continuous linear operator on $\mathcal{U}_\theta$ by

$$\Delta_E^Q := \Delta_G^Q + N^Q = \sum_{j=1}^{\infty} M^{Q+}_{\langle \cdot, e_j\rangle} D^+_{e_j} + \sum_{j=1}^{\infty} M^{Q-}_{\langle \cdot, e_j\rangle} D^-_{e_j},$$

where for $z \in N'$,

$$M^{Q-}_{\langle \cdot, z\rangle} = \sigma^{-1}(M_{\langle \cdot, z\rangle} \otimes I)\sigma, \quad M^{Q+}_{\langle \cdot, z\rangle} = \sigma^{-1}(I \otimes M_{\langle \cdot, z\rangle})\sigma,$$

and $M_{\langle \cdot, z\rangle}$ is the multiplication operator by $\langle \cdot, z\rangle$, see [18].

Theorem 6 *Let $\Xi \in \mathcal{U}_\theta$ and $t \in \mathbb{R}\backslash\{0\}$. Then for each $T \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$*

$$\lim_{s\to 0} \langle\!\langle T, \frac{S^Q_{t+s}\Xi - S^Q_t \Xi}{s} \rangle\!\rangle = \frac{1}{t} \langle\!\langle T, \Delta_E^Q S_t^Q \Xi \rangle\!\rangle.$$

Proof By Theorem 4 and Eq. (25) we have

$$\begin{aligned}
\lim_{s\to 0} \langle\!\langle T, \frac{S^Q_{t+s}\Xi - S^Q_t \Xi}{s} \rangle\!\rangle &= \lim_{s\to 0} \langle\!\langle \frac{1}{s} \left\{ F(1+\frac{s}{t}) \diamond \Gamma^Q(1+\frac{s}{t})(T) - T \right\}, S_t^Q \Xi \rangle\!\rangle \\
&= \lim_{s\to 0} \langle\!\langle F(1+\frac{s}{t}) \diamond \frac{1}{s}(\Gamma^Q(1+\frac{s}{t})(T) - T) \\
&+ \frac{1}{s}(F(1+\frac{s}{t}) - I) \diamond T, S_t^Q \Xi \rangle\!\rangle.
\end{aligned}$$

Since, we have

$$\lim_{s\to 0} \left\{ \frac{(1+\frac{s}{t})^{l+m} - 1}{s} \right\} = \frac{d}{ds} \left(1+\frac{s}{t}\right)^{l+m} \Big|_{s=0} = \frac{1}{t}(l+m),$$

for $T = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\Phi_{l,m})$, we get

$$\lim_{s\to 0} \frac{\Gamma^Q(1+\frac{s}{t})(T) - T}{s} = \sum_{l,m=0}^{\infty} \frac{1}{t}(l+m)\Xi_{l,m}(\Phi_{l,m}) = \frac{1}{t} N^Q T. \tag{34}$$

On the other hand using (21), we have

$$\lim_{s\to 0} \sigma(\frac{1}{s}\{F(1+\frac{s}{t}) - I\})(\xi, \eta)$$

$$
\begin{aligned}
&= \lim_{s\to 0}\frac{1}{s}\left\{\exp\left[\Big(\frac{(1+\frac{s}{t})^2-1}{2}\Big)(\langle\xi,\xi\rangle+\langle\eta,\eta\rangle)\right]-1\right\}\\
&= \frac{1}{t}\big(\langle\xi,\xi\rangle+\langle\eta,\eta\rangle\big)\\
&= \frac{1}{t}\sigma\big(\Xi_{0,2}(\tau)+\Xi_{2,0}(\tau)\big)(\xi,\eta).
\end{aligned}
$$

Then, from Eqs. (14) and (15) we get

$$
\begin{aligned}
\lim_{s\to 0}\langle\!\langle T,\frac{S^Q_{t+s}\Xi-S^Q_t\Xi}{s}\rangle\!\rangle &= \langle\!\langle \frac{1}{t}N^Q T+\frac{1}{t}(\Delta^Q_G)^*T, S^Q_t\Xi\rangle\!\rangle\\
&= \frac{1}{t}\langle\!\langle T,\Delta^Q_E S^Q_t\Xi\rangle\!\rangle.
\end{aligned}
$$

Which gives the desired statement.

Remark 3 Using Theorem 6, for $\Xi\in\mathcal{U}_\theta$, we have

$$
\lim_{s\to 0}\frac{S^Q_{e^s}\Xi-\Xi}{s}=\lim_{s\to 0}\left(\frac{e^s-1}{s}\right)\frac{S^Q_{(e^s-1+1)}\Xi-\Xi}{e^s-1}=\Delta^Q_E\Xi.
$$

This shows that $\{S^Q_{e^t}\}$ is a semigroup on $\mathcal{U}_\theta$ with infinitesimal generator Δ^Q_E. Hence, we deduce that $S^Q_{e^t}U_0$ is the unique solution of the Cauchy problem

$$
\frac{\partial}{\partial t}U_t=\Delta^Q_E U_t,\quad U_0\in\mathcal{U}_\theta.
$$

Theorem 7 *(Euler's theorem). Let $\Xi\in\mathcal{U}_\theta$. Then Ξ is $\lambda-$order homogeneous if and only if it satisfies the following* `QWN`*-Euler equation*

$$
\Delta^Q_E\Xi=\lambda\Xi\cdot \tag{35}
$$

Proof If Ξ is λ-order homogeneous, then by (26) we have

$$
\Delta^Q_E\Xi=\lim_{t\to 1}\frac{S^Q_t(\Xi)-S^Q_1(\Xi)}{t-1}=\lim_{t\to 1}\frac{t^\lambda-1}{t-1}\Xi=\lambda\Xi.
$$

Conversely, suppose that (35) is satisfied. Let $t\in\mathbb{R}\backslash\{0\}$. Put $G(t)=t^{-\lambda}S^Q_t(\Xi)$. Then, by (35) and Theorem 6 we get

$$
\begin{aligned}
\lim_{s\to 0}\frac{G(t+s)-G(t)}{s} &= \lim_{s\to 0}\frac{1}{s}\Big\{(t+s)^{-\lambda}S^Q_{t+s}(\Xi)-t^{-\lambda}S^Q_t(\Xi)\Big\}\\
&= \lim_{s\to 0}\frac{1}{s}\Big\{(t+s)^{-\lambda}(S^Q_{t+s}(\Xi)-S^Q_t(\Xi))\Big\}\\
&\quad+\lim_{s\to 0}\frac{1}{s}\Big\{(t+s)^{-\lambda}-t^{-\lambda}\Big\}S^Q_t(\Xi)\\
&= t^{-(\lambda+1)}\Delta^Q_E(S^Q_t(\Xi))-\lambda t^{-(\lambda+1)}S^Q_t(\Xi).
\end{aligned}\tag{36}
$$

Now, let $T\in\mathcal{L}(\mathcal{F}_\theta,\mathcal{F}^*_\theta)$. Then, by Theorem 4 and Theorem 6, we have

$$
\begin{aligned}
\langle\!\langle T,S^Q_t(\Delta^Q_E\Xi)\rangle\!\rangle &= \langle\!\langle F(t)\diamond\Gamma^Q(t)(T),\Delta^Q_E\Xi\rangle\!\rangle\\
&= \lim_{s\to 0}\langle\!\langle F(t)\diamond\Gamma^Q(t)(T),\frac{S^Q_{1+s}\Xi-\Xi}{s}\rangle\!\rangle\\
&= \lim_{s\to 0}\langle\!\langle T,\frac{S^Q_tS^Q_{1+s}\Xi-S^Q_t\Xi}{s}\rangle\!\rangle.
\end{aligned}
$$

But, by applying the QWN-Scaling transformation on $\Xi^{x,y}$, for $x,y\in N$, we can show that $S_u^Q S_v^Q = S_v^Q S_u^Q$ for all $u,v\in\mathbb{R}$. Then we get

$$\langle\!\langle T, S_t^Q(\Delta_E^Q \Xi)\rangle\!\rangle = \lim_{s\to 0}\langle\!\langle T, \frac{S_{1+s}^Q S_t^Q \Xi - S_t^Q \Xi}{s}\rangle\!\rangle.$$

Hence, using Theorem 6 , we obtain

$$\langle\!\langle T, S_t^Q(\Delta_E^Q \Xi)\rangle\!\rangle = \langle\!\langle T, \Delta_E^Q(S_t^Q \Xi)\rangle\!\rangle,$$

from which we deduce that

$$S_t^Q(\Delta_E^Q \Xi) = \Delta_E^Q(S_t^Q \Xi).$$

Therefore, using (35) we get

$$\Delta_E^Q(S_t^Q(\Xi)) = \lambda S_t^Q(\Xi).$$

Thus, from (36) we deduce that $G'(t)=0$ for all $t\in\mathbb{R}\backslash\{0\}$. In particular, $G(t) = G(1)$, i.e.,

$$t^{-\lambda} S_t^Q(\Xi) = S_1^Q(\Xi) = \Xi\cdot$$

From which we deduce the desired statement.

Remark 4 Euler's theorem remains valid in $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ where Δ_E^Q is replaced by $\widetilde{\Delta}_E^Q$ acting on $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ as follows

$$\widetilde{\Delta}_E^Q(\Xi) = \Xi_{-\tau}\diamond \Delta_E^Q(\Xi_\tau \diamond \Xi).$$

Corollary 1 *Let $\lambda\in\mathbb{N}$ and $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m})\in\mathcal{U}_\theta$ such that $\Delta_G^Q(\Xi)=0$. Then Ξ is $\lambda-$order homogeneous if and only if $\Xi = \sum_{l=0}^\lambda \Xi_{l,\lambda-l}(\kappa_{l,\lambda-l})$.*

Proposition 3 *Let $\Xi\in\mathcal{U}_\theta$ be a $\lambda-$order homogeneous operator such that $\Delta_G^Q(\Xi) = 0$. Then for each $\xi\in N$, $D_\xi^{\pm}(\Xi)$ are $(\lambda-1)-$order homogeneous and $(D_\xi^{\pm})^*(\Xi)$ are $(\lambda+1)-$order homogeneous.*

Proof We recall from [6] that, for any $\xi\in N$, the following identities hold true

$$D_\xi^+ \Xi_{l,m}(\kappa_{l,m}) = l\Xi_{l-1,m}(\xi\otimes^1 \kappa_{l,m})$$

$$D_\xi^- \Xi_{l,m}(\kappa_{l,m}) = m\Xi_{l,m-1}(\kappa_{l,m}\otimes_1 \xi)$$

$$(D_\xi^+)^* \Xi_{l,m}(\kappa_{l,m}) = \Xi_{l+1,m}(\xi\otimes\kappa_{l,m})$$

$$(D_\xi^-)^* \Xi_{l,m}(\kappa_{l,m}) = \Xi_{l,m+1}(\kappa_{l,m}\otimes\xi).$$

Then, if $\Xi = \sum_{l=0}^\lambda \Xi_{l,\lambda-l}(\kappa_{l,\lambda-l})$, we have

$$D_\xi^+\Xi = \sum_{l=0}^{\lambda-1}(l+1)\Xi_{l,\lambda-l-1}(\xi\otimes^1 \kappa_{l+1,\lambda-1-l}). \tag{37}$$

Thus, the fact $\Delta_G^Q(D_\xi^+\Xi) = D_\xi^+(\Delta_G^Q\Xi) = 0$ and identity (37) proves the statement for $D_\xi^+(\Xi)$ via Corollary 1. The others statements can be verified by slight modification.

Theorem 8 *Let $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{U}_\theta$. Then $\Xi_{l,m}(\kappa_{l,m})$ is $(l+m)-$order homogeneous if and only if $\Delta_G^Q(\Xi_{l,m}(\kappa_{l,m})) = 0$.*

Proof From (16) we have

$$\Delta_G^Q \Xi_{l,m}(\kappa_{l,m}) = l(l-1)\Xi_{l-2,m}(\tau \otimes^2 \kappa_{l,m}) + m(m-1)\Xi_{l,m-2}(\kappa_{l,m} \otimes_2 \tau). \quad (38)$$

Then by iterating (38) we get

$$(\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) =$$

$$\sum_{p=0}^{[l/2]} \sum_{p+q=k}^{[m/2]} \frac{l!m!k!}{(l-2p)!(m-2q)!p!q!} \Xi_{l-2p,m-2q}(\tau^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau^{\otimes q}). \quad (39)$$

On the other hand from (27), we obtain

$$S_t^Q \Xi_{l,m}(\kappa_{l,m}) = \sum_{p=0}^{[l/2]} \sum_{q=0}^{[m/2]} \frac{l!m!}{(l-2p)!(m-2q)!p!q!} t^{l+m-2p-2q} \big(\frac{1}{2}(t^2-1)\big)^{p+q}$$
$$\times \Xi_{l-2p,m-2q}(\tau^{\otimes p} \otimes^{2p} \kappa_{l,m} \otimes_{2q} \tau^{\otimes q}). \quad (40)$$

Then in view of (39), (40) becomes

$$S_t^Q \Xi_{l,m}(\kappa_{l,m})$$

$$= \sum_{k=0}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \big(\frac{1}{2}(t^2-1)\big)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m})$$
$$= t^{l+m} \Xi_{l,m}(\kappa_{l,m}) + \sum_{k=1}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \big(\frac{1}{2}(t^2-1)\big)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}).$$

It then follows that $\Xi_{l,m}(\kappa_{l,m})$ is $(l+m)-$homogeneous if and only if

$$\sum_{k=1}^{[(l+m)/2]} \frac{1}{k!} t^{l+m-2k} \big(\frac{1}{2}(t^2-1)\big)^k (\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) = 0. \quad (41)$$

Hence, using the fact that $\{P_k(X) = X^{l+m-2k}(X^2-1)^k;\ k = 1, 2, \cdots, [(l+m)/2]\}$ is a linearly independent family of polynomials, one can show that (41) holds if and only if

$$(\Delta_G^Q)^k \Xi_{l,m}(\kappa_{l,m}) = 0, \quad \forall k = 1, 2, \cdots, [(l+m)/2].$$

This implies in particular that $\Delta_G^Q \Xi_{l,m}(\kappa_{l,m}) = 0$. The converse is straightforward by Euler's theorem.

References

1. Accardi, L., Barhoumi, A., Ji, U.C.: Quantum Laplacians on Generalized Operators on Boson Fock space. Probability and Mathematical Statistics, Vol. 31, 1-24 (2011).
2. Accardi, L., Smolyanov, O.G.: On Laplacians and Traces. Conferenze del Seminario di Matematica dell'Università di Bari, Vol. 250, 1-28 (1993).
3. Accardi, L., Smolyanov, O.G.: Transformations of Gaussian measures generated by the Lévy Laplacian and generalized traces. Dokl. Akad. Nauk SSSR, Vol. 350, 5-8 (1996).
4. Barhoumi, A., Lanconelli, A., Rguigui, H.: QWN-Convolution operators with application to differential equations. Random Operators and Stochastic Equations, Vol. 22 (4), 195-211 (2014).
5. Barhoumi, A., Ouerdiane, H., Rguigui, H.: Stochastic Heat Equation on Algebra of Generalized Functions. Infinite Dimensional Analysis, Quantum Probability and Related Topics, Vol. 15, No. 4, 1250026 (18 pages) (2012).
6. Barhoumi, A., Ouerdiane, H., Rguigui, H.: QWN-Euler Operator And Associated Cauchy problem. Infinite Dimensional Analysis Quantum Probability and Related Topics, Vol. 15, No. 1, 1250004 (20 pages) (2012).
7. Barhoumi, A., Ouerdiane, H., Rguigui, H.: Generalized Euler heat equation. Quantum Probability and White Noise Analysis, Vol. 25, 99–116 (2010).
8. Gannoun, R., Hachaichi, R., Ouerdiane, H., Rezgi, A.: Un théorème de dualité entre espace de fonction holomorphes à croissance exponentielle. J. Funct. Anal., Vol. 171, 1-14 (2000).
9. Huang, Z.Y. , Hu, X.S., Wang, X.J.: Explicit forms of Wick tensor powers in general white noise spaces. IJMMS, Vol. 31, 413–420 (2002).
10. Gel'fand, I.M., Shilov, G.E.: Generalized Functions, Vol. I. Academic Press, Inc., New York (1968).
11. Grothaus, M., Streit, L.: Construction of relativistic quantum fields in the framework of white noise analysis. J. Math. Phys. Vol. 40, 5387–5405 (1999).
12. Ji, U.C., Obata, N.: Annihilation-derivative, creation-derivative and representation of quantum martingales. Commun. Math. Phys., Vol. 286, 751-775 (2009).
13. Ji, U.C., Obata, N., Ouerdiane, H.: Analytic characterization of generalized Fock space operators as two-variable entire function with growth condition. Infinite Dimensional Analysis Quantum Probability and Related Topics, Vol. 5, No 3, 395-407 (2002).
14. Kuo, H.H.: On Fourier transform of generalized Brownian functionals. J. Multivariate Anal. Vol. 12, 415-431 (1982).
15. Kuo, H.H.: The Fourier transform in white noise calculus. J. Multivariate Analysis, Vol. 31, 311-327 (1989).
16. Kuo, H.H.: White noise distribution theory. CRC press, Boca Raton (1996).
17. Liu, K., Yan, J.A.: Euler operator and Homogenoeus Hida distributions. Acta Mathematica Sinica, Vol. 10, No 4, (1994), 439-445.
18. Obata, N.: White noise calculus and Fock spaces. Lecture notes in Mathematics 1577, Spriger-Verlag (1994).
19. Ouerdiane, H., Rguigui, H.: QWN-Conservation Operator And Associated Differential Equation. Communication on stochastic analysis, Vol. 6, No. 3, 437-450 (2012).
20. Yan, J.A.: Products and Transforms of white noise Functionals. Appl. Math. Optim. Vol. 31, 137-153 (1995).
21. Potthoff, J., Yan, A.: Some results about test and generalized functionals of white noise. Proc. Singapore Prob. Conf. L.Y. Chen et al.(eds.) , 121-145 (1989).
22. Rguigui, H.: Quantum Ornstein-Uhlenbeck semigroups. Quantum Studies: Math. and Foundations, Vol. 2, 159-175 (2015).
23. Rguigui, H.: Quantum λ-potentials associated to quantum Ornstein - Uhlenbeck semigroups. Chaos, Solitons & Fractals, Vol. 73, 80-89, (2015).
24. Rguigui, H.: Characterization of the QWN-conservation operator. Chaos, Solitons & Fractals, Vol. 84, 41-48 (2016).

View publication stats